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One-Particle Subspaces in the Stochastic *XY* Model

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We study the spectrum of the generator H_β of the Glauber dynamics for a model of planar rotators on a lattice in the case of a high temperature $1/\beta$. We construct two so-called one-particle subspaces $\mathscr{H}_\pm$ for H_β and describe the spectrum of the generator in these subspaces. As a consequence we find time asymptotics of the correlations for the Glauber dynamics.

KEY WORDS: Plane rotators; stochastic dynamics; invariant subspaces; Dirichlet forms and operators.

1. INTRODUCTION AND DESCRIPTION OF THE MODEL

We study here stochastic (Glauber) dynamics for an infinite system of plane rotators in the high-temperature regime. This dynamics (given by a Markov semigroup) is constructed in such a way that a Gibbsian distribution μ for the system of plane rotators (the so-called XY model) is invariant with respect to the dynamics. By using this semigroup we can define a reversible stationary Markov process with the stationary distribution μ. Such an approach to the study of Gibbsian measures began with works of Dobrushin,[6] Holley,[7] and Holley and Stroock[8,9] and was developed extensively and intensively in the study of many models of classical and quantum statistical physics, quantum field theory, etc. For a review we refer the reader to refs. 3 and 12.

Let us denote by H the generator of the stochastic dynamics, i.e., the generator of corresponding Markov semigroup acting in the space $L^2(\Omega, \mu)$ (Ω is the phase space of our system, see below). We are interested in the spectral properties of the self-adjoint operator H, namely in the structure

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of the lower branches of its spectrum. We find two invariant subspaces $(\mathscr{H}_+, \mathscr{H}_-)$ for H (so-called one-particle subspaces) and describe in detail the spectrum of H in these subspaces. We also show that the remaining part of the spectrum of H lies above that spectrum.

From the general point of view we deal here with the wider idea which consists in establishing a quasiparticle picture for the operator H which controls the dynamics of a system with an infinite number of components with a local, translation-invariant interaction.

The first step is to find the so-called one-particle subspaces $\mathscr{H}_1,..., \mathscr{H}_k$ for the operator H. These subspaces are invariant with respect to the operator H and are cyclic with respect to a group of translation operators U_s. We assume that this group is isomorphic to the lattice $\mathbb{Z}^d$. There exist unitary mappings

$$V_j: \quad H_j \to L^2(T^d, d\lambda)$$

where T^d denotes the d-dimensional torus, i.e., the group of characters of the group $\mathbb{Z}^d$. These mappings transform the operators $U_s | \mathscr{H}_i$ and $H | \mathscr{H}_i$ into the multiplication operators by the functions

$$\exp i(\lambda, s), \qquad \lambda \in T^d, \quad s \in \mathbb{Z}^d$$

and

$$m_i(\lambda), \qquad \lambda \in T^d$$

respectively. The spaces $\mathscr{H}_1,..., \mathscr{H}_k$ describe states of quasiparticles (elementary excitations). The function $m_i(\lambda)$ is the dispersion of a particle of the ith kind, i.e., $m_i(\lambda)$ is the energy of this particle as a function of its quasimomentum $\lambda \in T^d$.

The present work is devoted to this first step in the picture of the quasiparticle representation. The next step in this picture is a description of the whole system as a "free gas of quasiparticles" as well as an investigation of their "bounded states." This step is described in detail in refs. 14 and 16.

The problem of constructing the one-particle subspaces is well developed in the case of lattice models of quantum Euclidean fields with weak interaction in discrete space-time.[14]

In the case of continuous space with continuous time the one-particle spaces for the $P(\phi)_2$ model were constructed in the pionering work of Glimm, Jaffe, and Spencer with the help of the cluster expansion for the corresponding Markov field.[17] In the present paper (as well as in ref. 15) the spectral analysis of Markov field generator with continuous time is performed without using the cluster expansion of the whole field. We believe that this is an essential technical achivement in the model.

Let us go to exact definitions. The lattice used in our model will be the d-dimensional square lattice $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. For $k = (k^{(1)},..., k^{(d)}) \in \mathbb{Z}^d$ we shall use the norm $|k| = |k^{(1)}| + \cdots + |k^{(d)}|$. The single-spin space of our model is the unit circle T in the plane. We identify the circle with $[0, 2\pi]$, where 0 and 2π are considered to be the same point. We use the normalized Haar measure $[1/(2\pi)]\, dx = d\nu_0(x)$ on T. The phase space of the model we consider is the infinite-dimensional torus

$$\Omega = T^{\mathbb{Z}^d} \ni x = (x_k)_{k \in \mathbb{Z}^d}$$

endowed with the product topology and with the natural product measure $d\mu_0(x) = \mathsf{X}_{k \in \mathbb{Z}^d}\, d\nu_0(x_k)$. Given $\Lambda \in \mathbb{Z}^d$,

$$\Omega \ni x \mapsto x_\Lambda = (x_k)_{k \in \Lambda} \in T^\Lambda$$

denotes the natural projection, and the symbol $C^p_\Lambda(\Omega)$ denotes the set of functions on Ω of the form

$$\Omega \ni x \mapsto f(x_\Lambda) \in \mathbb{C}$$

where f runs over the set $C^p(T^\Lambda)$ of continuously differentiable (up to the order p) complex-valued functions on T^Λ. We shall use also the set of finitely based functions

$$\mathscr{F}C^p(\Omega) = \bigcup_{\Lambda :\, |\Lambda| < \infty} C^p_\Lambda(\Omega)$$

The Hamiltonian of the *XY* model is formally given by

$$U(x) = -\sum_{\langle kj \rangle} \cos(x_k - x_j), \qquad x \in \Omega \tag{1.1}$$

where $\langle kj \rangle$ denotes, as usual, a pair of nearest neighbors of $\mathbb{Z}^d$. The equilibrium state of the *XY* model at the inverse temperature $\beta > 0$ is defined as the Gibbsian modification μ_β of the measure μ_0 by using the Hamiltonian $\beta U(x)$. Formally speaking, it means that μ_β is a probability measure on Ω of the form

$$d\mu_\beta(x) = \frac{1}{Z_\beta}\, e^{-\beta U(x)}\, d\mu_0(x)$$

For sufficiently small β such a measure is uniquely defined and has many good properties (see, e.g., ref. 5). Actually, the choice $\beta > 0$ is done for the

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simplicity of our formulas. With the same success we can consider also (small) $\beta < 0$.

For the construction of the stochastic dynamics in the XY model we start with the classical Dirichlet form which corresponds to μ_β. This form is defined for $u, v \in \mathscr{F}C^1(\Omega)$ as

$$\mathscr{E}_\beta(u, v) = \int_\Omega \sum_{k \in \mathbb{Z}^d} \frac{\partial u}{\partial x_k} \frac{\partial \bar{v}}{\partial x_k} \, d\mu_\beta \tag{1.2}$$

Let us introduce for any $k \in \mathbb{Z}^d$ the operator ∇_k in $L^2(\mu_\beta)$ on the domain $\mathscr{F}C^\infty(\Omega)$ by the formula

$$L^2(\mu_\beta) \supset \mathscr{F}C^\infty(\Omega) \ni f \mapsto \nabla_k f = \frac{\partial f}{\partial x_k} \in \mathscr{F}C^\infty(\Omega) \tag{1.3}$$

Then the definition of the Gibbs measure μ_β gives the following representation for the adjoint operator ∇_k^*:

$$\nabla_k^* f(x) = -\nabla_k f(x) - b_k(x) f(x) \tag{1.4}$$

with

$$b_k(x) = -\beta \sum_{l:\, |l-k| = 1} \sin(x_k - x_l) \tag{1.5}$$

We introduce the differential operator H_β on the domain $\mathscr{D}(H_\beta)$ in $L^2(\mu_\beta)$ by the formula

$$(H_\beta f)(x) = -\sum_{k \in \mathbb{Z}^d} \frac{\partial^2 f(x)}{\partial x_k^2} - \sum_{k \in \mathbb{Z}^d} b_k(x) \frac{\partial f(x)}{\partial x_k} \tag{1.6}$$

By the definition of $\mathscr{E}_\beta$ and (1.4) we have

$$(H_\beta u, v)_{L^2(\mu_\beta)} = \mathscr{E}_\beta(u, v), \qquad u, v \in \mathscr{F}C^2(\Omega)$$

The operator H_β is a symmetric operator in $L^2(\mu_\beta)$. As was shown in refs. 1 and 2, H_β is an essentially self-adjoint operator, i.e., the closure of H_β in $L^2(\mu_\beta)$ is a self-adjoint operator. For this closure we keep the same notation. Then the stochastic dynamics in the XY model is defined via the Markov semigroup

$$T_t^\beta = e^{-tH_\beta}, \qquad t \geq 0$$

This semigroup generates a symmetric diffusion process on the infinite-dimensional torus Ω with the invariant distribution μ_β. Such a diffusion can be constructed directly as the solution to an infinite-dimensional system of stochastic differential equations with the state space Ω.[10]

Finally we mention another interpretation of the operator H_β as a "ground-state Hamiltonian" of an infinite system of quantum particles attached to each point of the lattice $\mathbb{Z}^d$ and moving on T. To this end we introduce a formal potential of interaction between particles in the form

$$V_{\text{form}}(x) = \frac{\beta}{2} \sum_{k \in \mathbb{Z}^d} \left\{ \frac{\beta}{2} \left[\sum_{l:\, |l-k|=1} \sin(x_k - x_l) \right]^2 - \sum_{l:\, |l-k|=1} \cos(x_k - x_l) \right\}$$

and a formal Schrödinger operator of the system as

$$H_{\text{form}} = -\sum_{k \in \mathbb{Z}^d} \frac{\partial^2}{\partial x_k^2} + V_{\text{form}}$$

For a rigorous definition of the quantum dynamics in the model considered we will use the following "ground-state renormalization scheme" (see, e.g., refs. 11, 4, and 14). For any finite $\Lambda \subset \mathbb{Z}^d$ we introduce the Hamiltonian of the system in the volume Λ as a self-adjoint operator in the physical Hilbert space

$$L^2(T^\Lambda, \nu_\Lambda), d\nu_\Lambda(x_\Lambda) = \mathop{\times}_{k \in \Lambda} d\nu_0(x_k)$$

as follows

$$H_\Lambda = -\sum_{k \in \Lambda} \frac{\partial^2}{\partial x_k^2} + V_\Lambda(x_\Lambda)$$

where V_Λ is defined similarly to V_{form}, but the summation is extended over $k, l \in \Lambda$. It is easy to check that the function

$$\phi_\Lambda(x_\Lambda) = \frac{1}{N_\Lambda} \exp\left\{ \frac{\beta}{2} \sum_{\langle kl \rangle \subset \Lambda} \cos(x_k - x_l) \right\}$$

is the normalized ground state of H_Λ. Here N_Λ is the normalizing factor, $H_\Lambda \phi_\Lambda = E_\Lambda \phi_\Lambda$, $E_\Lambda = \inf \operatorname{sp}(H_\Lambda)$. Let us introduce the unitary map

$$L^2(T^\Lambda, \nu_\Lambda) \ni \psi \mapsto \phi_\Lambda^{-1} \psi \in L^2(T^\Lambda, \mu_\Lambda)$$

where $\mu_\Lambda = \phi_\Lambda^2 \nu_\Lambda$. This map transfers the shifted Hamiltonian $H_\Lambda - E_\Lambda$ into the renormalized Hamiltonian $H_{\Lambda,\text{ren}}$ given by the Dirichlet operator associated with the measure μ_Λ:

$$H_{\Lambda,\text{ren}} = -\sum_{k\in\Lambda} \frac{\partial^2}{\partial x_k^2} + \beta \sum_{k\in\Lambda} \left[\sum_{l\in\Lambda:\,|l-k|=1} \sin(x_k - x_l) \right] \frac{\partial}{\partial x_k} \tag{1.7}$$

Under our assumptions about β the measures μ_Λ converge to the limit measure μ_β when $\Lambda \to \to \mathbb{Z}^d$ in the local weak topology. This means that for any $f \in \mathscr{F}C(\Omega)$ we have

$$\int_{T^\Lambda} f\, d\mu_\Lambda \to \int_\Omega f\, d\mu_\beta, \qquad \Lambda \to \mathbb{Z}^d$$

Having in the mind this picture, we can consider the operator H_β as a renormalized operator of the infinite quantum lattice system in the ground state corresponding to H_{form}.

Note in conclusion that our constructions and results immediately can be extended to a more general class of models in which the interaction (1.1) has the form

$$U(x) = \sum_{\alpha,k,j} J_{k-j} P_\alpha(x_k)\, P_\alpha(x_j)$$

where P_α, $\alpha \in A$, are trigonometric polynomials, and the index α runs over a finite set A, and J_s, $s \in \mathbb{Z}^d$, is a finite support function on the lattice: $J_s = 0$, $|s| > R$ for some $R \in \mathbb{N}$. Moreover, apparently everything can be extended to the case of unbounded spin systems with a polynomial self-interaction potential. This will be the subject of a forthcoming paper. Let us remark that analogous results with respect to the Ising model were obtained by one of us in ref. 15.

2. RESULTS

Let $\{U_j \mid j \in \mathbb{Z}^d\}$ be the unitary group of translations acting in $L^2(\mu_\beta)$ by the formula

$$(U_j f)(x) = f(\tau_j x)$$

where $(\tau_j x)_k = x_{k-j}$ is the shift of a configuration $x \in \Omega$ by the vector $j \in \mathbb{Z}^d$. This group obviously commutes with H_β. There is another unitary group of (continuous) symmetry:

$$\{V_q \mid q \in T\}, \qquad (V_q f)(x) = f(x+q)$$

where $(x+q)_k = x_k + q$. This group also commutes with H_β. The generator M of the unitary group V_q has the following form:

$$(Mf)(x) = i \sum_{k \in \mathbb{Z}^d} \frac{\partial f(x)}{\partial x_k}, \qquad f \in \mathscr{F}C^1(\Omega) \tag{2.1}$$

Remark 2.1. With the physical interpretation of H_β as the energy operator of vacuum excitations [see (1.7)] the operator M is the charge operator. Then the spaces $\mathscr{H}_+$ and $\mathscr{H}_-$ constructed below describe particles with the charge $+1$ and -1, respectively.

Finally, there exists a unitary involution

$$(Jf)(x) = f(-x)$$

which also commutes with H_β.

Let us denote by $\mathscr{M}$ the set of finite support multi-indices:

$$\mathscr{M} \ni n = (n(k))_{\mathbb{Z}^d}, \qquad n(k) \in \mathbb{Z}, \qquad n(k) = 0, \qquad |k| > N(n)$$

For $n \in \mathscr{M}$ we put $\|n\|^2 = \sum_{k \in \mathbb{Z}^d} n^2(k)$. In the case $\beta = 0$ the orthonormal basis in the space $L^2(\mu_0)$ is formed with the functions

$$e_n(x) = \bigtimes_{k \in \mathbb{Z}^d} e^{in(k)x_k}, \qquad n \in \mathscr{M}$$

These functions are the eigenfunctions of the operator H_0:

$$H_0 e_n = \sum_{k \in \mathbb{Z}^d} n^2(k)\, e_n, \qquad n \in \mathscr{M}$$

From this we see that the operator H_0 has the following eigenspaces: $\mathscr{H}_0^0 = \mathbb{C}$ (so-called vacuum space) corresponding to the eigenvalue 0, and

$$\mathscr{H}_+^0 = \left\{ \sum_{k \in \mathbb{Z}^d} c_k e^{ix_k} \right\}, \qquad \mathscr{H}_-^0 = \left\{ \sum_{k \in \mathbb{Z}^d} c_k e^{-ix_k} \right\}$$

which correspond to the eigenvalue 1 and are called the one-particle subspaces. These spaces are also proper for the operator M with the eigenvalues ± 1, respectively. The spectrum of H_0 in the orthogonal complement to $\mathscr{H}_0^0 \oplus \mathscr{H}_+^0 \oplus \mathscr{H}_-^0$ consists of the eigenvalues $E_k = 2, 3,...$ of infinte multiplicity.

The perturbed operator H_β has obviously the same vacuum space. We clarify the situation with one-particle subspaces of H_β in the following theorem.

Theorem 2.1. For small enough β there exist two orthogonal subspaces $\mathscr{H}_\pm \subset L^2(\mu_\beta)$ which are invariant with respect to the operators H_β, U_j, and M. The operator J transforms each of them to other:

$$J\mathscr{H}_\pm = \mathscr{H}_\mp$$

The spectra of the operators

$$H_\pm = H_\beta|_{\mathscr{H}_\pm}$$

in the subspaces $\mathscr{H}_\pm$ coincide and lie in some small neighborhood of the value 1. More precisely, there are unitary maps

$$V_\pm: \mathscr{H}_\pm \to L^2(T^d, d\lambda)$$

[T^d is the d-dimensional torus with the Haar measure

$$d\lambda = dv_0(\lambda^{(1)}) \cdots dv_0(\lambda^{(d)}), \qquad \lambda = (\lambda^{(1)},..., \lambda^{(d)}) \in T^d]$$

which transform the operators $H_\pm$ into a multiplication operator

$$(\tilde{H}f)(\lambda) = \tilde{m}(\lambda) f(\lambda), \qquad f \in L^2(T^d, d\lambda)$$

and the operators U_j, $j \in \mathbb{Z}^d$, into the operators

$$(\tilde{U}_j f)(\lambda) = \exp\{i(\lambda, j)\} f(\lambda), \qquad (\lambda, j) = \sum_{s=1}^{d} \lambda^{(s)} j^{(s)}$$

The function $\tilde{m}(\lambda)$ is an analytic function in some complex neighborhood W of the torus T^d (this neighborhood will be described below) and has the form

$$\tilde{m}(\lambda) = 1 - 2\beta \sum_{s=1}^{d} \cos \lambda^{(s)} + \kappa(\lambda)$$

where $|\kappa(\lambda)| < C_1 \beta^2$, $\lambda \in W$, with an absolute constant C_1.

The spaces $\mathscr{H}_\pm$ consist of eigenvectors of the operator M with eigenvalues ± 1, respectively.

The spectrum of H_β in the orthogonal complement to $\mathscr{H}_0 \oplus \mathscr{H}_+ \oplus \mathscr{H}_-$ lies above the value $2 - C_2\beta$, where C_2 is an absolute constant and thus is isolated from the spectrum of H_β in $\mathscr{H}_\pm$.

Remark 2.2. As usual, we say that a closed subspace $\mathscr{H}' \subset \mathscr{H}$ is an invariant subspace of a unbounded operator H in $\mathscr{H}$ if there exists a set $\mathscr{L} \subset \mathscr{H}' \cap D(H)$ which is dense in $\mathscr{H}'$ and such that $H\mathscr{L} \subset \mathscr{H}'$.

Theorem 2.1 gives as a consequence an asymptotics for the decay of correlations of the Markov process $\xi(t) = \{\xi_k(t) \mid k \in \mathbb{Z}^d\}$, $t \geqslant 0$, generated by the semigroup T_t^β. Let $\mathscr{P}$ denote the probability distribution of this process. For any $k_0 \in \mathbb{Z}^d$ and given $f, g \in C(T)$, $\int f(q)\, e^{iq}\, dv_0(q) \neq 0$, $\int g(q)\, e^{iq}\, dv_0(q) \neq 0$ or $\int f(q)\, e^{-iq}\, dv_0(q) \neq 0$, $\int g(q)\, e^{-iq}\, dv_0(q) \neq 0$, we consider the correlation

$$\begin{aligned} I_{f,g}^{(k_0)}(t) &\equiv \langle f(\xi_{k_0}(0)), g(\xi_{k_0}(t)) \rangle_{\mathscr{P}} \\ &= \langle f(\xi_{k_0}(0))\, g(\xi_{k_0}(t)) \rangle_{\mathscr{P}} - \langle f(\xi_{k_0}(0)) \rangle_{\mathscr{P}} \langle g(\xi_{k_0}(t)) \rangle_{\mathscr{P}} \end{aligned}$$

Here $\langle \cdot \rangle_{\mathscr{P}}$ means the expectation with respect to the measure $\mathscr{P}$.

Theorem 2.2. The following asymptotics is true

$$I_{f,g}^{(k_0)}(t) = \frac{C}{t^{d/2}}\, e^{-mt}(1 + o(1)), \qquad t \to \infty$$

where

$$m = \min_{\lambda \in T^d} \tilde{m}(\lambda)$$

and $C = C(f, g)$ is a constant depending on f and g.

3. THE CONSTRUCTION OF THE ONE-PARTICLE SUBSPACES $\mathscr{H}_+$ AND $\mathscr{H}_-$

3.1. Preliminary Considerations

Let us introduce the space L of functions on Ω having the form

$$f(x) = \sum_{n \in \mathscr{M}} f_n e_n(x) \tag{3.1}$$

with the condition that $\| f \|_L \equiv \sum_{n \in \mathscr{M}} |f_n| < \infty$. Note that under this condition the series (3.1) converges absolutely and uniformly on Ω. Evidently L is isomorphic to the space $l_1(\mathscr{M})$. It is obvious that L is a dense subset of $L^2(\mu_\beta)$ and

$$\forall f \in L, \qquad \| f \|_{L^2(\mu_\beta)} \leqslant \| f \|_L \tag{3.2}$$

For any bounded operator B in L we denote by $B_{n,n'}$ its matrix elements in the basis $\{e_n \mid n \in \mathcal{M}\}$:

$$Be_n = \sum_{n' \in \mathcal{M}} B_{n,n'} e_{n'}$$

Note that the coefficients of f in the decomposition (3.1) are transformed by B as follows:

$$(Bf)_n = \sum_{n'} B_{n',n} f_{n'}$$

It is clear that for the operator norm of B in L we have

$$\|B\|_L = \sup_n \sum_{n' \in \mathcal{M}} |B_{n,n'}|$$

Further, we will use the following simple lemma.

Lemma 3.1. Let B be a symmetric operator in $L^2(\mu_\beta)$ such that $BL \subset L$ and the restriction $B|_L$ is a bounded operator in L. Then B is a bounded operator in $L^2(\mu_\beta)$, and we have the following inequality between operator norms:

$$\|B\|_{L^2(\mu_\beta)} \leqslant \|B\|_L$$

For the proof see ref. 15.

By using (1.5), (1.6) it is easy to calculate that

$$H_\beta e_n = \|n\|^2 e_n + \beta \sum_{n'} W_{n,n'} e_{n'} \tag{3.3}$$

with

$$W_{n,n'} = \sigma n(u(b)) \qquad \text{if} \quad n' = n + m_b^\sigma \quad \text{for some } m_b^\sigma \tag{3.4}$$

and $W_{n,n'} = 0$ otherwise. Here $b = (u, v)$ is an oriented edge of the lattice, u is the beginning and v is the end of b, $\sigma = \pm 1$, and $m_b^\sigma \in \mathcal{M}$ is the function

$$m_b^\sigma(k) = \begin{cases} \sigma, & k = u \\ -\sigma, & k = v \\ 0, & k \neq u, v \end{cases}$$

We denote by $\tilde{L}$ a subspace of L which consists of finite sums of the form (3.1). From (3.3), (3.4) we see that $H_\beta \tilde{L} \subset \tilde{L}$.

Let us introduce the following notations: for any $n \in \mathcal{M}$ we put (in addition to $\|n\|$)

$$\sigma(n) = \sum_{k \in \mathbb{Z}^d} n(k), \qquad |n| = \sum_{k \in \mathbb{Z}^d} |n(k)|$$

Further,

$$\mathcal{M}_1^{\pm} = \{n \in \mathcal{M} \mid \sigma(n) = \pm 1, |n| = 1\}$$

$$\mathcal{M}_{>1}^{\pm} = \{n \in \mathcal{M} \mid \sigma(n) = \pm 1, |n| > 1\}$$

$$\mathcal{M}_{\text{rest}} = \{n \in \mathcal{M} \mid |\sigma(n)| \neq 1, |n| > 1\}$$

Let $L_1^{\pm}$, $L_{>1}^{\pm}$, L_{rest} be the corresponding subspaces in L:

$$L_1^{\pm} = \left\{ f \in L \,\middle|\, f = \sum_{n \in \mathcal{M}_1^{\pm}} f_n e_n \right\}, \quad \text{etc.}$$

Then we have the decomposition

$$L = L_0 + L_1^+ + L_1^- + L_{>1}^+ + L_{>1}^- + L_{\text{rest}}$$

where $L_0 = \mathbb{C}$. Introduce the spaces

$$L^{\pm} = L_1^{\pm} + L_{>1}^{\pm}$$

Lemma 3.2. The spaces L_0, $L^{\pm}$, and L_{rest} are invariant with respect to H_β.

Proof. The spaces $L^{\pm}$ are proper for the operator M with the eigenvalues ± 1, respectively. Because H_β commutes with M, we have $H_\beta \tilde{L}^{\pm} \subset L^{\pm}$, where $\tilde{L}^{\pm} = L^{\pm} \cap \tilde{L}$. The space L_{rest} admits the decomposition $L_{\text{rest}} = \sum_{r \neq \pm 1} L^{(r)}$, where

$$L^{(r)} = \left\{ f = \sum_{n \in \mathcal{M}_{\text{rest}}, \sigma(n) = r} f_n e_n \right\}$$

Each $L^{(r)}$ is proper for M and $H_\beta \tilde{L}^{(r)} \subset \tilde{L}^{()}$. From this $H_\beta \tilde{L}_{\text{rest}} \subset \tilde{L}_{\text{rest}}$ ($\tilde{L}_{\text{rest}} = L_{\text{rest}} \cap \tilde{L}$). ∎

Furthermore, the spaces $L_0 = \mathbb{C}$, L_β^+, L_β^-, $L_{\text{rest},\beta} \subset L^2(\mu_\beta)$ [the closures of L^+, L^-, L_{rest} in the norm of $L^2(\mu_\beta)$] are mutually orthogonal. Thus we have to study H_β in each subspace L_β^+, L_β^-, and $L_{\text{rest},\beta}$ separately.

3.2. The Estimate of the Spectrum of $H_\beta|_{L_{\mathrm{rest},\beta}}$

Lemma 3.3. The operator

$$H_\beta|_{\tilde{L}_{\mathrm{rest}}} = \tilde{H}_{\beta,\mathrm{rest}} \qquad \text{in} \quad \tilde{L}_{\mathrm{rest}}$$

is invertible and its inverse $\tilde{H}^{-1}_{\beta,\mathrm{rest}}$ is bounded (in the norm of L) and extends as a bounded operator on L_{rest} with the following norm estimate:

$$\|\tilde{H}^{-1}_{\beta,\mathrm{rest}}\|_L < \frac{1}{2(1-4\beta d)} \tag{3.5}$$

Corollary 3.1. From the estimate (3.5) and Lemma 3.1 it follows that

$$\|(H|_{L_{\mathrm{rest},\beta}})^{-1}\|_{L^2(\mu_\beta)} < \frac{1}{2(1-4\beta d)} \tag{3.6}$$

and therefore the spectrum of H_β in $L_{\mathrm{rest},\beta}$ lies above $2-8\beta d$.

Proof of Lemma 3.3. It is convenient for us to identify a function $f(x)=\sum_{n\in\mathcal{M}_{\mathrm{rest}}} f_n e_n(x)$ from L_{rest} with the sequence of coefficients $\{f_n \mid n\in\mathcal{M}_{\mathrm{rest}}\}$. We will use the same identification also for other spaces considered. Then by using formulas (3.3), (3.4) we can transfer the operator $H_{\beta,\mathrm{rest}}$ from functions on corresponding sequences. Keeping in mind such transfer, we write

$$H_{\beta,\mathrm{rest}} = H^0_{\beta,\mathrm{rest}} + \beta W_{\mathrm{rest}}$$

where

$$\begin{aligned} (H^0_{\beta,\mathrm{rest}} f)_n &= \|n\|^2 f_n, \qquad n\in\mathcal{M}_{\mathrm{rest}} \\ (W_{\mathrm{rest}} f)_n &= \beta \sum_{n'\in\mathcal{M}_{\mathrm{rest}}} W_{n',n} f_{n'}, \qquad n\in\mathcal{M}_{\mathrm{rest}} \end{aligned} \tag{3.7}$$

Further,

$$H^{-1}_{\beta,\mathrm{rest}} = (H^0_{\beta,\mathrm{rest}})^{-1}(E+V)^{-1}$$

where $V=W_{\mathrm{rest}}(H^0_{\beta,\mathrm{rest}})^{-1}$ and E is the identity operator in L_{rest}. Then

$$(Vf)_n = \beta \sum_{n'} W_{n',n} \frac{1}{\|n'\|^2} f_{n'}$$

From this

$$\begin{aligned}
\|Vf\|_L &\leqslant \beta \sum_{n,n'} |W_{n,n}| \frac{1}{\|n'\|^2} |f_{n'}| \\
&= \beta \sum_{n'} \frac{|f_{n'}|}{\|n'\|^2} \sum_{(\sigma,b),u(b)\in \operatorname{supp} n'} |W_{n',n'+m_b^\sigma}| \\
&< 2\beta \sum_{n'} \frac{|f_{n'}|}{\|n'\|^2} \sum_{b,u(b)\in \operatorname{supp} n'} |n'(u(b))| \\
&< 4\beta d \sum_{n'\in \mathcal{N}_{\text{rest}}} \frac{|n'|}{\|n'\|^2} |f_{n'}| \leqslant 4\beta d \|f\|_L
\end{aligned}$$

This gives

$$\|(E+V)^{-1}\|_L < \frac{1}{1-4\beta d}$$

and hence

$$\|H_{\beta,\text{rest}}^{-1}\|_L \leqslant \|(H_{\beta,\text{rest}}^0)^{-1}\|_L \|(E+V)^{-1}\|_L < \frac{1}{2(1-4\beta d)}$$

The lemma is proved. ∎

3.3. The Study of H_β in the Space L^+

Lemma 3.4. There are two closed subspaces $\mathscr{L}_1^+ \subset L^+$, $\mathscr{L}_{>1}^+ \subset L^+$ such that

$$\mathscr{L}_1^+ + \mathscr{L}_{>1}^+ = L^+$$

and each of them is invariant with respect to U_j, $j \in \mathbb{Z}^d$, and the operator H_β.

Proof. The decomposition

$$L^+ = L_1^+ + L_{>1}^+$$

generates the representation of $H^+ = H_\beta|_{L^+}$ by the matrix

$$\begin{pmatrix} H_{00}^+ & H_{01}^+ \\ H_{10}^+ & H_{11}^+ \end{pmatrix}$$

in which $H_{00}^+ : \tilde{L}_1^+ \to \tilde{L}_1^+$, $H_{01}^+ : \tilde{L}_{>1}^+ \to \tilde{L}_1^+$, etc.

We find the spaces $\mathscr{L}_{\mathrm{I}}^{+}$ and $\mathscr{L}_{>\mathrm{I}}^{+}$ in the form of the graphs of some operators

$$S:\quad L_{\mathrm{I}}^{+} \to L_{>\mathrm{I}}^{+}$$

and

$$T:\quad L_{>\mathrm{I}}^{+} \to L_{\mathrm{I}}^{+}$$

namely,

$$\mathscr{L}_{\mathrm{I}}^{+} = \{u + Su \mid u \in L_{\mathrm{I}}^{+}\} \tag{3.8}$$

$$\mathscr{L}_{>\mathrm{I}}^{+} = \{v + Tv \mid v \in L_{>\mathrm{I}}^{+}\} \tag{3.9}$$

The requirement of the invariance of the spaces $\mathscr{L}_{\mathrm{I}}^{+}$ and $\mathscr{L}_{>\mathrm{I}}^{+}$ is equivalent to the relations

$$S(H_{00}^{+} + H_{01}^{+} S) = H_{10}^{+} + H_{11}^{+} S$$

and

$$T(H_{10}^{+} T + H_{11}^{+}) = H_{00}^{+} T + H_{01}$$

These relations [under the assumption that $(H_{11}^{+})^{-1}$ exists] can be rewritten as

$$S = (H_{11}^{+})^{-1} SH_{00}^{+} + (H_{11}^{+})^{-1} SH_{01}^{+} - (H_{11}^{+})^{-1} H_{10}^{+} \tag{3.10}$$

$$T = H_{00}^{+} T(H_{11}^{+})^{-1} - TH_{10}^{+} T(H_{11}^{+})^{-1} + H_{01}^{+}(H_{11}^{+})^{-1} \tag{3.11}$$

The existence of $(H_{11}^{+})^{-1}$ and an estimate of its norm will be obtained below. In the following we use the notation $\mathscr{L}(E_1 \to E_2)$ for the space of bounded linear operators from a normed space E_1 into a normed space E_2 equipped with the operator norm.

Lemma 3.5. Let us suppose that $\beta d < 1/100$. Then there exist unique solutions S and T to the equations (3.10), (3.11) with small norms:

$$\|S\|_{\mathscr{L}(L_{\mathrm{I}}^{+} \to L_{>\mathrm{I}}^{+})} \leqslant 11\beta d$$

$$\|T\|_{\mathscr{L}(L_{>\mathrm{I}}^{+} \to L_{\mathrm{I}}^{+})} \leqslant 11\beta d$$

Proof. We can consider the right-hand side of Eq. (3.10) as a map $\mathscr{F}$ of the space $\mathscr{L}(L_{\mathrm{I}}^{+} \to L_{>\mathrm{I}}^{+})$ into itself. We shall show that there exists a ball

$$\{S \in \mathscr{L}(L_{\mathrm{I}}^{+} \to L_{>\mathrm{I}}^{+}) \mid \|S\|_{\mathscr{L}(L_{\mathrm{I}}^{+} \to L_{>\mathrm{I}}^{+})} \leqslant \kappa\}$$

which is transformed by $\mathscr{F}$ into itself, and that $\mathscr{F}$ acts on this ball as a contraction.

For simplicity of notation, for a bounded linear map $B: E_1 \to E_2$ between subspaces $E_1, E_2 \subset L$ we will denote by $\|B\|$ the norm of B in $\mathscr{L}(E_1 \to E_2)$.

For the study of $\mathscr{F}$ we need to estimate the norms $\|H_{10}^+\|$, $\|H_{01}^+\|$, $\|H_{00}^+\|$, and $\|(H_{11}^+)^{-1}\|$. By using (3.7) just as in the proof of Lemma 3.7, we come to the following results:

1. $\|(H_{11}^+)^{-1}\| < \frac{1}{2}(1-4\beta d)^{-1}$
2. $\|H_{00}^+\| < 1 + 2\beta d$
3. $\|H_{01}^+\| < 4\beta d$
4. $\|H_{10}^+\| < 2\beta d$

From this, for any $S \in \mathscr{L}(L_1^+ \to L_{>1}^+)$, we have

$$\|\mathscr{F}S\| < \frac{1+2\beta d}{2-8\beta d}\|S\| + \frac{4\beta d}{2-8\beta d}\|S\|^2 + \frac{2\beta d}{2-8\beta d}$$

Then for any κ which satisfies the inequality

$$\frac{1+2\beta d}{2-8\beta d}\kappa + \frac{4\beta d}{2-8\beta d}\kappa^2 + \frac{2\beta d}{2-8\beta d} < \kappa$$

the mapping $\mathscr{F}$ transforms the ball

$$Y_\kappa = \{S \mid \|S\| < \kappa\}$$

into itself. Furthermore, if in addition

$$\frac{1+2\beta d}{2-8\beta d} + \frac{8\beta d\kappa}{2-8\beta d} < 1$$

then $\mathscr{F}$ acts insde Y_κ as a contraction. For $\beta d \leqslant 1/100$ we put $\kappa = 11\beta d$. It is easy to check that then both inequalities are fulfilled. Lemma 3.5 is proved, and as a result the existence of subspaces $\mathscr{L}_1^+$ and $\mathscr{L}_{>1}^+$ in the form (3.8), (3.9) is proved also. From the uniqueness of the solutions to Eqs. (3.10), (3.11) it follows that S and T commute with operators U_j, $j \in \mathbb{Z}^d$. The latter implies the invariance of $\mathscr{L}_1^+$ and $\mathscr{L}_{>1}^+$ with respect to the group U_j, $j \in \mathbb{Z}^d$.

In order to show the decomposition $L^+ = \mathscr{L}_1^+ + \mathscr{L}_{>1}^+$, we should get for every $f = f_1 + f_{>1} \in L^+, f_1 \in L_1^+, f_{>1} \in L_{>1}^+$ the decomposition

$$f = u + Su + v + Tv, \qquad u \in L_1^+, \quad v \in L_{>1}^+ \tag{3.12}$$

with $u + Tv = f_1$, $v + Su = f_{>1}$. Then

$$v - STv = f_{>1} - Sf_1$$

and

$$u - TSu = f_1 - Tf_{>1}$$

Thus

$$v = (E_{>1} - ST)^{-1} (f_{>1} - Sf_1)$$

$$u = (E_1 - TS)^{-1} (f_1 - Tf_{>1})$$

Here $E_1, E_{>1}$ are unit operators in subspaces L_1^+ and $L_{>1}^+$, respectively. From this we get the decomposition (3.12). ∎

Our next aim is an estimate of the operator norms in L of the operators $H_\beta|_{\mathscr{L}_1^+}$ and $(H_\beta|_{\mathscr{L}_{>1}^+})^{-1}$.

Lemma 3.6. The following estimates are true (for $\beta d < 1/100$):

$$\|H_\beta|_{\mathscr{L}_1^+}\| \leqslant 1 + 15\beta d$$

$$\|(H_\beta|_{\mathscr{L}_{>1}^+})^{-1}\| \leqslant \frac{1}{2 - 31\beta d}$$

Proof. For $u \in L_1^+$ we have $H_\beta(u + Su) = \bar{u} + S\bar{u}$, where $\bar{u} = (H_{00}^+ + H_{01}^+ S)\, u$. Then using the inequality $\|u\|_L \leqslant \|u + Su\|_L$ we obtain

$$\begin{aligned} \|H_\beta(u + Su)\|_L &\leqslant (1 + \|S\|)\, \|\bar{u}\|_L \\ &\leqslant (1 + \|S\|)(\|H_{00}^+\| + \|H_{01}^+\| \cdot \|S\|)\, \|u\|_L \\ &\leqslant (1 + \|S\|)(\|H_{00}^+\| + \|H_{01}^\pm\| \cdot \|S\|)\, \|u + Su\|_L \end{aligned}$$

From here and the estimates 2 and 3 of the previous lemma we find, for $\beta d < 1/100$,

$$\|H_\beta|_{\mathscr{L}_1^+}\| \leqslant (1 + 11\beta d)(1 + 2\beta d + 44(\beta d)^2) \leqslant 1 + 15\beta d$$

By using similar arguments we have

$$\|(H_\beta)^{-1}|_{\mathscr{L}^+_{>1}}\| \leqslant (1+\|T\|)\ \|(H^+_{10}T+H^+_{11})^{-1}\|$$

Further,

$$(H^+_{11}+H^+_{10}T)^{-1}=(H^+_{11})^{-1}\,(E+H^+_{10}T(H^+_{11})^{-1})^{-1}$$

Thus, as before, from estimates 1 and 4 we find

$$\|(H^+_{11}+H^+_{10}T)^{-1}\| \leqslant \frac{1}{2(1-4\beta d)}\left[1-22(\beta d)^2\frac{1}{2(1-4\beta d)}\right]^{-1}$$
$$\leqslant [2-8\beta d-22(\beta d)^2]^{-1}<(2-9\beta d)^{-1}$$

Finally

$$\|(H_\beta)^{-1}|_{\mathscr{L}^+_1} \leqslant \frac{1+11\beta d}{2-9\beta d} \leqslant \frac{1}{2-31\beta d} \quad \blacksquare$$

Let us introduce the spaces $\mathscr{H}^+_1$ and $\mathscr{H}^+_{>1}$ as the closures in $L^2(\mu_\beta)$ of the spaces $\mathscr{L}^+_1$ and $\mathscr{L}^+_{>1}$, respectively. Evidently these spaces are invariant with respect to the operators H_β, U_j, $j\in\mathbb{Z}^d$, and are proper for the operator M with the eigenvalue 1.

From the estimates of Lemmas 3.6 and 3.1 it follows that the spectrum of the restriction $H_\beta|_{\mathscr{H}^+_1}$ lies below $1+15\beta d$ and the spectrum of $H_\beta|_{\mathscr{H}^+_{>1}}$ lies above $2-31\beta d$. Hence for $\beta d<100$ these spectra do not overlap. The latter implies that the spaces $\mathscr{H}^+_1$ and $\mathscr{H}^+_{>1}$ are orthogonal in $L^2(\mu_\beta)$.

Similar considerations and assertions are true for the spaces $\mathscr{H}^-_1$ and $\mathscr{H}^-_{>1}$, which we obtain in the same way starting with the decomposition of the space L^-.

As a result we have constructed the invariant subspaces $\mathscr{H}_+=\mathscr{H}^+_1$ and $\mathscr{H}_-=\mathscr{H}^-_1$ for H_β and U_j, $j\in\mathbb{Z}^d$, and have proved that the spectrum of the remaining part of H_β lies above the spectrum of H_β in these subspaces. This gives the proof of the first part of Theorem 2.1. In the next section we will study the spectrum of H_β in $\mathscr{H}_\pm$ in more detail.

4. THE SPECTRUM OF H_β IN $\mathscr{H}_+$ AND $\mathscr{H}_-$

Let us denote

$$e^+_k(x_k)=e^{ix_k}, \qquad k\in\mathbb{Z}^d, \quad x_k\in T$$

and introduce a basis in $\mathscr{L}_1^+$ of the form

$$v_k = e_k^+ + Se_k^+, \qquad k \in \mathbb{Z}^d$$

Evidently $U_j v_k = v_{k+j}$, $k, j \in \mathbb{Z}^d$. Any operator B in $\mathscr{L}_1^+$ which commutes with U_j, $j \in \mathbb{Z}^d$, has matrix elements $B_{k,j} = B_{k-j}$, $k, j \in \mathbb{Z}^d$, that depend on the difference $k-j$ (and as a result generates a convolution operator on coefficients of decompositions with respect to the basis). In particular, because the operator $H_\beta|_{\mathscr{L}_1^+} = H_1^+$ commutes with U_j, $j \in \mathbb{Z}^d$, its matrix elements in this basis have the form

$$H_1^+ v_k = \sum_l m(k-l)\, v_l \tag{4.1}$$

where m: $\mathbb{Z}^d \to \mathbb{C}$. For this function on the lattice we shall give an explicit expression.

We have $H_1^+ v_k = g_k + Sg_k$, where

$$g_k = \sum_l \left[(H_{00}^+)_{k-l}\, e_l^+ + (H_{01}^+ S)_{k-l}\, e_l^+ \right]$$

Further, from this

$$\begin{aligned} H_1^+ v_k &= \sum_l \left[(H_{00}^+ + H_{01}^+ S)_{k-l}\, e_l^+ + (H_{00}^+ + H_{01}^+ S)_{k-l}\, e_l^+ \right] \\ &= \sum_l (H_{00}^+ + H_{01}^+ S)_{k-l}\, v_l \end{aligned}$$

Thus

$$m(k-l) = (H_{00}^+)_{k-l} + (H_{01}^+ S)_{k-l} \tag{4.2}$$

It follows from the definition that

$$(H_{00}^+)_k = \begin{cases} 1, & k=0 \\ -\beta, & |k|=1 \\ 0, & |k|>1 \end{cases}$$

Let $S_{k,n}$, $k \in Z$, $n \in \mathscr{M}_{>1}^+$, be the matrix of the operator S: $L_1^+ \to L_{>1}^+$.

Lemma 4.1. The following representation holds:

$$S_{k,n} = R_{k,n} \left(\tfrac{1}{2}\right)^{d\{k \cup \operatorname{supp} n\}}$$

where

$$\sup_k \sum_n |R_{k,n}| \leqslant 11\beta d$$

and d_A is the length of the minimal connected subgraph $G \subset \mathbb{Z}^d$ such that $A \subset [G]$, $[G]$ is the set of vertices of G.

Corollary 4.1. From this lemma follows the estimate

$$|(H_{01}^+ S)_{k,j}| \leqslant 22\beta^2 d(\tfrac{1}{2})^{|k-j|}, \qquad k, j \in \mathbb{Z}^d$$

Indeed,

$$|(H_{01}^+ S)_{k,j}| = \left| \sum_{n, |n|>1} (H_{01}^+)_{n,k} S_{j,n} \right|$$
$$\leqslant \beta \sum_{n=\delta_k + m_h^a} |n(u(b))| \cdot |R_{j,n}| (\tfrac{1}{2})^{d_{\{j \cup \operatorname{supp} n\}}}$$

We have obviously $k \in \operatorname{supp} n$ (otherwise $|n| = 1$) and hence $|k-j| \leqslant d_{\{j \cup \operatorname{supp} n\}}$, as well as $|n(u(b))| \leqslant 2$. Thus

$$|(H_{01}^+ S)_{k,j}| \leqslant 2\beta(\tfrac{1}{2})^{|k-j|} \sup_j \sum_n |R_{j,n}| \leqslant 22\beta^2 d(\tfrac{1}{2})^{|k-j|}$$

Proof. Let us consider the space $\mathscr{A} \subset \mathscr{L}(L_1^+ \to L_{>1}^+)$ of operators Q whose matrix elements $Q_{k,n}$ satisfy the estimate

$$|||Q||| \equiv \sup_k \sum_n |Q_{k,n}| \, 2^{d_{\{k \cup \operatorname{supp} n\}}} < \infty$$

In other words, we can rewrite

$$Q_{k,n} = R_{k,n}(\tfrac{1}{2})^{d_{\{k \cup \operatorname{supp} n\}}}$$

where $\sup_k \sum_n |R_{k,n}| < \infty$.

We shall show that the operator S constructed in the previous section belongs to $\mathscr{A}$ and $|||S||| \leqslant 11\beta d$.

To this end we would like to show that the map $\mathscr{F}$ transforms $\mathscr{A}$ into itself and that on the ball

$$\hat{Y} = \{ Q \in \mathscr{A} \mid |||Q||| < 11\beta d \}$$

the mapping $\mathscr{F}$ acts as a contraction (with respect to the norm $|||\cdot|||$). Because $\|Q\| \leqslant |||Q|||$ the ball $\hat{Y}$ lies inside the ball

$$Y = \{Q \mid \|Q\| \leqslant 11\beta d\}$$

Thus the fixed point $\hat{Q}$ of the operator $\mathscr{F}$ in $\hat{Y}$ coincides with the unique fixed point S of $\mathscr{F}$ in Y. This gives the assertion of the lemma.

We need to estimate the norms of all operators in Eq. (3.10).

We have

$$(H_{11}^{+})^{-1} = (H_{11}^{0})^{-1} \sum_{p=0}^{\infty} V^{p}$$

where [see (3.7)]

$$(Vf)_n = \beta \sum_{b, \sigma:\, n + m_b^\sigma \in \mathscr{M}_{>1}^{+}} \sigma \frac{(n + m_b^\sigma)(u(b))}{\|n + m_b^\sigma\|^2} f(n + m_b^\sigma) \tag{4.3}$$

Then

$$(V^p f)_n = \beta^p \sum \prod_{s=1}^{p} \sigma_s \frac{\prod_{t=1}^{p} (n + \sum_{s=1}^{t} m_{b_s}^{\sigma_s})(u(b_t))}{\prod_{t=1}^{p} \|n + \sum_{s=1}^{t} m_{b_s}^{\sigma_s}\|^2} f\left(n + \sum_{s=1}^{p} m_s^{\sigma_s}\right)$$

where the summation is extended over all ordered collections $(b_1, \sigma_1), \ldots, (b_p, \sigma_p)$ of pairs (b_s, σ_s), $s = 1, \ldots, p$, so that $n + \sum_{s=1}^{t} m_{b_s}^{\sigma_s} \in \mathscr{M}_{>1}^{+}$ for any $t = 1, \ldots, p$. From all this we obtain

$$|(V^p Q)_{k,n}| \leqslant \beta^p \sum \prod_{t=1}^{p} \frac{|(n + \sum_{s=1}^{t} m_{b_s}^{\sigma_s})(u(b_t))|}{\|n + \sum_{s=1}^{t} m_{b_s}^{\sigma_s}\|^2}$$
$$\times |R_{k, n + \sum_{s=1}^{p} m_{b_s}^{\sigma_s}}| \left(\frac{1}{2}\right)^{d\{k \cup \operatorname{supp}(n + \sum_{s=1}^{p} m_{b_s}^{\sigma_s})\}} \tag{4.4}$$

Note that for any collection $\{(b_s, \sigma_s),\ s = 1, \ldots, p\}$ in the sum (4.4) the following inequality is fulfilled:

$$p + d_{\{k \cup \operatorname{supp}(n + \sum_{s=1}^{p} m_{b_s}^{\sigma_s})\}} \geqslant d_{\{k \cup \operatorname{supp} n\}} \tag{4.5}$$

Indeed the beginning $u(b_p)$ of the last edge b_p belongs to $\operatorname{supp} n'$, where $n' = n + \sum_{s=1}^{p} m_{b_s}^{\sigma_s}$. The point $u(b_{p-1})$ belongs to $\operatorname{supp}(n' - m_{b_p}^{\sigma_p}) \subset \operatorname{supp} n' \cup b_p$, the point $u(b_{p-2})$ belongs to $\operatorname{supp}(n' - m_{b_p}^{\sigma_p} - m_{b_{p-1}}^{\sigma_{p-1}}) \subset \operatorname{supp}(n' \cup b_p \cup b_{p-1})$. Doing so, we get for any $q < p$

$$u(b_{p-q-1}) \subset \operatorname{supp}(n' \cup b_p \cup \cdots \cup b_{p-q})$$

Moreover, $\operatorname{supp} n \subset \operatorname{supp} n' \cup b_1 \cup \cdots \cup b_p$. From this, for any connected graph G such that $k \cup \operatorname{supp} n' \subset [G]$ the graph $G'' = G \cup b_1 \cup \cdots \cup b_p$ is also connected and $\operatorname{supp} n \subset [G']$. This gives the inequality (4.5).

Thus we have the estimate

$$|(V^p Q)_{k,n}| \leqslant (2\beta)^p \left(\tfrac{1}{2}\right)^{d(k \cup \operatorname{supp} n)} \tilde{R}^{(p)}_{k,n} \tag{4.6}$$

where

$$\tilde{R}^{(p)}_{k,n} = \sum \frac{|(n' - \sum_{s=2}^{p} m^{\sigma_s}_{b_s})(u(b_1))|}{\|n' - \sum_{s=2}^{p} m^{\sigma_s}_{b_s}\|^2} \times \frac{|(n' - \sum_{s=3}^{p} m^{\sigma_s}_{b_s})(u(b_2))|}{\|n' - \sum_{s=3}^{p} m^{\sigma_s}_{b_s}\|^2} \cdots \frac{|n'(b_p^{\sigma_p})|}{\|n'\|^2} |R_{k,n'}|$$

In the latter expression the summation is extended over $(\sigma_1, b_1),..., (\sigma_p, b_p)$, n' which satisfy the conditions $|n' - \sum_{s=q}^{p} m^{\sigma_s}_{b_s}| > 1$, $q = 2,..., p$, $|n'| > 1$, and $n' - \sum_{s=1}^{p} m^{\sigma_s}_{b_s} = n$.

Now we perform the summation over (σ_1, b_1) for fixed $(\sigma_2, b_2),...,$ (σ_p, b_p), n'. Then we do so over (σ_2, b_2) with the others fixed, and so on. As a result we get

$$\sum_n |\tilde{R}^{(p)}_{k,n}| \leqslant (4d)^p \sum_{n'} |R_{k,n}| \leqslant (4d)^p \, |||Q|||$$

Finally,

$$\begin{aligned}
&|((H^+_{11})^{-1} QH^+_{00})_{k,n}| \\
&\quad\leqslant \frac{1}{\|n\|^2} \sum_{p=0}^{\infty} |(V^p QH^+_{00})_{k,n}| \\
&\quad\leqslant \frac{1}{2} \sum_{p=0}^{\infty} |(V^p Q)_{k,n}| + \beta \sum_{p=0}^{\infty} \sum_{e \in \mathbb{Z}^d,\, |e| = 1} |(V^p Q)_{k+e,n}| \\
&\quad\leqslant \frac{1}{2}\left(\frac{1}{2}\right)^{d(k \cup \operatorname{supp} n)} \left[\sum_{p=0}^{\infty} (2\beta)^p |\tilde{R}^{(p)}_{k,n}| + 2\beta \sum_{p,e} (2\beta)^p |\tilde{R}^{(p)}_{k+e,n}| \right]
\end{aligned}$$

Then

$$|||(H^+_{11})^{-1} QH^+_{00}||| \leqslant \frac{1}{2} \sum_{p=0}^{\infty} (8\beta d)^p (1 + 4\beta d) \, |||Q||| = \frac{1 + 4\beta d}{2(1 - 8\beta d)} \, |||Q|||$$

Repeating the previous considerations, we have

$$|||(H_{11}^{+})^{-1} QH_{01}^{+} Q||| \leqslant \frac{\beta d}{1-8\beta d} |||Q|||^2$$

and

$$|||(H_{11}^{+})^{-1} H_{10}^{+}||| \leqslant \frac{4\beta d}{1-8\beta d}$$

Thus we find that

$$|||\mathscr{F} Q||| \leqslant \frac{1+4\beta d}{2(1-8\beta d)} |||Q||| + \frac{\beta d}{1-8\beta d} |||Q|||^2 + \frac{4\beta d}{1-8\beta d}$$

As in the proof of Lemma 3.5, the latter estimate shows that for $\beta d < 1/100$ the operator $\mathscr{F}$ transforms $\hat{Y}$ into itself and acts on $\hat{Y}$ as a contraction.

Remark 4.1. In a similar way it can be shown that the matrix elements of the operator T satisfy the same estimate

$$|T_{n,k}| \leqslant 11\beta d(\tfrac{1}{2})^{d(k, \operatorname{supp} n)}$$

From Corollary 4.1 and formula (4.2) it follows that

$$m(k) = m_0(k) + m_1(k), \qquad k \in \mathbb{Z}^d \tag{4.7}$$

where

$$m_0(k) = \begin{cases} 1, & k=0 \\ -\beta, & |k|=1 \\ 0, & |k|>1 \end{cases}$$

and

$$|m_1(k)| \leqslant 22\beta^2 d(\tfrac{1}{2})^{|k|}$$

Now we need to introduce an orthonormal basis in $\mathscr{H}_1^+$. Let us consider the Gramm matrix $D_\beta = (d_{k,j})_{k,j \in \mathbb{Z}^d}$ with the elements

$$d_{k,j} = d_{k-j} = (v_k, v_j)_\beta = ((e_k^+ + Se_k^+), (e_j^+ + Se_j^+))_\beta$$

where $(\cdot, \cdot)_\beta \equiv (\cdot, \cdot)_{L^2(\mu_\beta)}$.

Lemma 4.2. The elements of the Gramm matrix D_β have the form

$$d_{k-j} = \delta_{k,j} + a_{k-j}, \qquad \text{where} \quad |a_{k-j}| \leqslant C\beta(\tfrac{1}{2})^{|k-j|} \tag{4.8}$$

with an absolute constant C.

We give the proof of this lemma in the Appendix.

The matrix D_β generates a convolution operator in the space $l_2(\mathbb{Z}^d)$ acting by the formula

$$l_2(\mathbb{Z}^d) \ni u = \{u(k) \mid k \in \mathbb{Z}^d\} \mapsto (D_\beta u)(k) = \sum_{j \in \mathbb{Z}^d} d_{k-j}\, u(j)$$

Lemma 4.3. There exist operators $D_\beta^{1/2}$ and $D_\beta^{-1/2}$ which act in $l_2(\mathbb{Z}^d)$ as convolutions:

$$(D_\beta^{1/2} u)(k) = \sum_{j \in \mathbb{Z}^d} d_{k-j}^{(1/2)}\, u(j)$$

$$(D_\beta^{-1/2} u)(k) = \sum_{j \in \mathbb{Z}^d} d_{k-j}^{(-1/2)} u(j)$$

The functions $d_k^{(\pm 1/2)}$ have the form

$$d_k^{(\pm 1/2)} = \delta_{k,0} + b_k^{\pm}, \qquad k \in \mathbb{Z}^d$$

where

$$|b_k^{\pm}| \leqslant C\beta(3/4)^{|k|}$$

with an absolute constant C.

This lemma is proved in the Appendix.

Introduce now a new basis in $\mathscr{H}_+$ by using the operator $D_\beta^{-1/2}$. For any $k \in \mathbb{Z}^d$ we put

$$w_k = \sum_{j \in \mathbb{Z}^d} d_{k-j}^{(-1/2)} v_j$$

From this definition it follows immediately that $\{w_k \mid k \in \mathbb{Z}^d\}$ is an orthonormal basis in $\mathscr{H}_+$ and $U_j w_k = w_{k+j}$. The operator H_1^+ acts in the basis $\{w_k \mid k \in \mathbb{Z}^d\}$ as a convolution:

$$H_1^+ w_k = \sum_{j \in \mathbb{Z}^d} \hat{m}(k-j)\, w_j$$

where

$$\hat{m}(k) = d^{(-1/2)} * m * d^{(1/2)}(k) = m(k), \qquad k \in \mathbb{Z}^d$$

Here $*$ denotes the discrete convolution operator (due to the commutativity of this operation we have $\hat{m} = m$).

Let us introduce the unitary map

$$V_+: \quad \mathscr{H}_+ \to L^2(T^d, d\lambda)$$

given by the relation

$$(V_+ w_k)(\lambda) = e^{i(k, \lambda)}, \qquad k \in \mathbb{Z}^d, \quad \lambda \in T^d$$

This map transforms the translation operators U_j, $j \in \mathbb{Z}^d$, into the operator

$$(\tilde{U}_j f)(\lambda) = e^{i(j, \lambda)} f(\lambda), \qquad f \in L^2(T^d, d\lambda)$$

and the convolution operator H_1^+ into the operator $\tilde{H}_1^+$ given by the formula

$$(\tilde{H}_1^+ f)(\lambda) = \tilde{m}(\lambda)\, f(\lambda), \qquad l \in T^d$$

Here $\tilde{m}(\lambda) = \sum_{k \in \mathbb{Z}^d} m(k)\, e^{i(k, \lambda)}$, $\lambda \in T^d$. From (4.2) and Corollary 4.2 we have

$$\tilde{m}(\lambda) = 1 - 2\beta \sum_{s=1}^{d} \cos \lambda^{(s)} + \kappa(\lambda) \tag{4.9}$$

where $\kappa(\lambda)$ is an analytic function in the strip

$$W = \{ |\mathrm{Im}\, \lambda^{(s)}| < \log 3/2 \mid s = 1,..., d\} \subset \mathbb{C}^d/\mathbb{Z}^d$$

Here $\mathbb{C}^d/\mathbb{Z}^d$ is a complex manifold obtained by the factorization of $\mathbb{C}^d$ with respect to the shift group

$$\mathbb{C}^d \ni z = (z^{(1)},..., z^{(d)}) \mapsto z + 2\pi k, \qquad k \in \mathbb{Z}^d$$

Due to (4.7) we have the estimate

$$|\kappa(\lambda)| \leqslant 22d4^d\beta^2, \qquad \lambda \in W \tag{4.10}$$

Because the spectrum of H_1^+ coincides with the set of values of the function $\tilde{m}$ from (4.9) and (4.10) we get that this spectrum is included in $[1 - 2d\beta - 11d4^d\beta^2,\ 1 + 2d\beta + 11d4^d\beta^2]$.

In the case of the space $\mathscr{H}_-$ we arrive at the same function $\tilde{m}(\lambda)$, as follows from the unitary equivalence of H_1^+ and H_1^- given by

$$JH_1^+J^{-1} = H_1^-$$

where J is the unitary involution introduced in Section 2. Thus Theorem 2.1 is proved completely. ▌

5. ASYMPTOTICS OF THE CORRELATIONS

In this section we give the proof of Theorem 2.1. Let f be a function on T with the Fourier decomposition

$$f(q) = \sum_{m \in \mathbb{Z}} c_m e^{imq}, \qquad q \in T$$

We suppose that $\sum_m |c_m| < \infty$. Let $F_f^{(k_0)}$, $k_0 \in \mathbb{Z}^d$, denote the following cylinder function on Ω:

$$F_f^{(k_0)}(x) = f(x_{k_0}), \qquad x \in \Omega$$

It is clear that $F_f^{(k_0)} \in L$ and

$$F_f^{(k_0)}(x) = c_0 + c_1 e_{\delta_{k_0}}(x) + c_{-1} e_{-\delta_{k_0}}(x) + \hat{F}_f^{(k_0)}(x)$$

where $\delta_{k_0} \in \mathscr{M}_1^+$, $\delta_{k_0}(k) = \delta_{k,k_0}$, $k \in \mathbb{Z}^d$, $e_{\delta_{k_0}} \in L_1^+$, $e_{-\delta_{k_0}} \in L_1^-$, and $\hat{F}_f^{(k_0)} \in L_{\text{rest}}$.
Further,

$$e_{\delta_{k_0}} = h_{k_0,1}^+ + h_{k_0,>1}^+, \qquad h_{k_0,1}^+ \in \mathscr{L}_1^+, \qquad h_{k_0,>1}^+ \in \mathscr{L}_{>1}^+$$

By the formulas that follow after (3.12) we have

$$h_{k_0,1}^+ = (E_1 - TS)^{-1} e_{\delta_{k_0}} + S(E_1 - TS)^{-1} e_{\delta_{k_0}} \in \mathscr{L}_1^+$$
$$h_{k_0,>1}^+ = -(E_{>1} - ST)^{-1} Se_{\delta_{k_0}} - T(E_{>1} - ST)^{-1} Se_{\delta_{k_0}} \in \mathscr{L}_{>1}^+$$

Let us consider the orthogonal decompostion

$$h_{k_0,1}^+ = \sum_{k \in \mathbb{Z}^d} q_k^{(k_0)} w_k \tag{5.1}$$

Lemma 5.1. The coefficients $q_k^{(k_0)}$ have the following representation:

$$q_k^{(k_0)} = \delta_{k,k_0} + \hat{q}_k^{(k_0)}, \qquad k \in \mathbb{Z}^d$$

where

$$|\hat{q}_k^{(k_0)}| \leqslant C\beta\tau^{|k-k_0|} \tag{5.2}$$

with an absolute constant C and some $\tau \in (0, 1)$.

The proof of this lemma is given in the Appendix.

From the representation (5.1) and (5.2) it follows that the function $\tilde{h}^+_{k_0,1} \equiv V_+ h^+_{k_0,1}$ is an analytic function on the torus T^d and $\tilde{h}^+_{k_0,1}(\lambda) \neq 0$, $\lambda \in T^d$.

For $e_{-\delta_{k_0}}$ we can construct a similar decomposition

$$e_{-\delta_{k_0}} = h^-_{k_0,1} + h^-_{k_0,>1}$$

and $\tilde{h}^-_{k_0,1} = V_- h^+_{k_0,1}$ is again an analytic function on the torus T^d. It is easy to see that $h^-_{k_0,1} = Jh^+_{k_0,1}$ and therefore $\tilde{h}^+_{k_0,1}(\lambda) = \bar{\tilde{h}}^-_{k_0,1}(\lambda)$.

As a result we have the following decomposition:

$$F_f^{(k_0)} = c_0 + c_1 h^+_{k_0,1} + c_{-1} h^-_{k_0,1} + c_1 h^+_{k_0,>1} + c_{-1} h^-_{k_0,>1} + \hat{F}_f^{(k_0)} \tag{5.3}$$

Note that $c_0 = \langle F_f^{(k_0)} \rangle_\beta$, where $\langle \cdot \rangle_\beta$ means the expectation with respect to μ_β.

A similar decomposition is true for the function $G_g^{(k_0)}(x) = g(x_{k_0})$, $x \in \Omega$:

$$G_g^{(k_0)} = \langle G_g^{(k_0)} \rangle_\beta + b_1 h^+_{k_0,1} + b_{-1} h^-_{k_0,1} + b_1 h^+_{k_0,>1} + b_{-1} h^-_{k_0,>1} + \hat{G}_g^{(k_0)} \tag{5.4}$$

where $\{b_s \mid s \in \mathbb{Z}\}$ are the Fourier coefficients of g.

It follows from the definition of the stochastic dynamics in our model that

$$\langle f(\xi_{k_0}(0))\, \bar{g}(\xi_{k_0}(t)) \rangle_{\mathscr{P}} = (F_f^{(k_0)}, T_t^\beta G_g^{(k_0)})_{L^2(\mu_\beta)}$$

where $T_t^\beta = e^{-tH_\beta}$, $t \geqslant 0$. Using (5.3) and (5.4), we obtain

$$\begin{aligned}
(F_f^{(k_0)}, T_t^\beta G_g^{(k_0)})_{L^2(\mu_\beta)} = {} & \langle F_f^{(k_0)} \rangle_\beta \langle \bar{G}_g^{(k_0)} \rangle_\beta \\
& + c_1 \bar{b}_1 (h^+_{k_0,1}, e^{-tH_1^+} h^+_{k_0,1})_{\mathscr{H}_1^+} \\
& + c_{-1} \bar{b}_{-1} (h^-_{k_0,1}, e^{-tH_1^-} h^-_{k_0,1})_{\mathscr{H}_1^-} \\
& + c_1 \bar{b}_1 (h^+_{k_0,>1}, e^{-tH^+_{>1}} h^+_{k_0,>1})_{\mathscr{H}^+_{>1}} \\
& + c_{-1} \bar{b}_{-1} (h^-_{k_0,>1}, e^{-tH^-_{>1}} h^-_{k_0,>1})_{\mathscr{H}^-_{>1}} \\
& + (\hat{F}_f^{(k_0)}, e^{-tH_{\text{rest}}} \hat{G}_g^{(k_0)})_{\mathscr{H}_{\text{rest}}}
\end{aligned} \tag{5.5}$$

Because the spectra of operators $H^{\pm}_{>1}$, H_{rest} are located above $2-31\beta d$ (see Lemma 3.6), we find that the last three terms in (5.5) have an estimate of the form const $\cdot\, e^{-t(2-31\beta d)}$.

Now we need to find the asymptotics of the second and third terms in (5.5). We have

$$(h^{+}_{k_0,1}, e^{-tH^{+}_{1}} h^{+}_{k_0,1})_{\mathscr{H}^{+}_{1}} = \int_{T^d} |\tilde{h}^{+}_{k_0,1}(\lambda)|^2\, e^{-t\tilde{m}(\lambda)}\, d\lambda \tag{5.6}$$

By using (4.9) it is not hard to see that the function $\tilde{m}(\lambda)$ has a unique point λ_0 of nondegenerate absolute minimum (lying near $\lambda=0$). Applying the Laplace method to (5.6), we get the following asymptotic equality:

$$\int_{T^d} |\tilde{h}^{+}_{k_0,1}(\lambda)|^2 \exp[-t\tilde{m}(\lambda)]\, d\lambda$$
$$= \frac{|\tilde{h}^{+}_{k_0,1}(\lambda_0)|^2}{(2\pi)^{3d/2}\, t^{d/2} (\text{Det } A)^{1/2}} \exp[-t\tilde{m}(\lambda_0)]\, (1+o(1)), \qquad t\to\infty$$

where A is the Hessian of the function $m(\lambda)$ at λ_0. The third term has similar asymptotics. From the stated asymptotics and (5.5) we get finally

$$\langle f(\xi_{k_0}(0)), g(\xi_{k_0}(t))\rangle_{\mathscr{P}}$$
$$= \frac{(c_1 \bar{b}_1 + c_{-1} \bar{b}_{-1})}{(2\pi)^{3d/2}\, t^{d/2} (\text{Det } A)^{1/2}} |\tilde{h}^{+}_{k_0,1}(\lambda_0)|^2 \exp[-t\tilde{m}(\lambda_0)]\, (1+o(1)), \quad t\to\infty$$

Theorem 2.2 is proved. ∎

APPENDIX

A.1. The Proof of Lemma 4.2

The elements of the Gramm matrix are

$$d_{k-j} = (e_k, e_j)_\beta + (Se_k, e_j)_\beta + (e_k, Se_j)_\beta + (Se_k, Se_j)_\beta$$

Let us consider each term separately.

1. $(e_k, e_k)_\beta = 1$ for any $k\in\mathbb{Z}^d$ and

$$|(e_k, e_j)_\beta| \leqslant (C\beta)^{|k-j|}, \qquad k, j\in\mathbb{Z}^d, \quad C\beta<1$$

where $C=C(d)$ is an absolute constant. This estimate follows from a standard consideration of the high-temperature regime; see, e.g., ref. 13.

If $C\beta < 1/2$, then

$$(C\beta)^{|k-j|} \leqslant 2C\beta(\tfrac{1}{2})^{|k-j|}, \qquad k \neq j \tag{A.1}$$

2. $(Se_k, e_j)_\beta = \sum_n S_{k,n}(e_n, e_j)_\beta$. Again it can be shown[13] that

$$|(e_n, e_j)_\beta| \leqslant (C\beta)^{\rho(\text{supp } n, j)} \leqslant (\tfrac{1}{2})^{\rho(\text{supp } n, j)}$$

where $\rho(A, j)$ is the distance between the set A and the point j. Using these estimates and Lemma 4.1, we arrive at the estimate

$$|(Se_k, e_j)_\beta| \leqslant \sum_n (\tfrac{1}{2})^{d|\text{supp } n \cup k|} |R_{k,n}| (\tfrac{1}{2})^{\rho(\text{supp } n, k)} \leqslant 11\beta(\tfrac{1}{2})^{|k-j|}$$

In a similar way we find

$$|(e_k, Se_j)_\beta| \leqslant 11\beta d(\tfrac{1}{2})^{|k-j|}$$

3. $|(Se_k, Se_n)_j| \leqslant (11\beta)^2 (1/2)^{|k-j|}$. Here we have used again the inequality

$$|(e_n; e_{n'})_\beta| \leqslant (C\beta)^{\rho(\text{supp } n, \text{ supp } n')}$$

where $\rho(A, B)$ denotes the distance between the sets A and B. From these estimates (4.8) follows immediately. ∎

A.2. The Proof of Lemma 4.3

It follows from Lemma 4.2 that the Gramm matrix D_β has a decomposition $D_\beta = E - A$, where E is the unity operator in $l_2(\mathbb{Z}^d)$ and A is a convolution operator. Let us consider the power decomposition

$$(1-t)^{1/2} = 1 - \sum_{p=1}^{\infty} \alpha_p t^p, \qquad \alpha_p > 0, \quad p \in \mathbb{N}$$

Then

$$D_\beta^{1/2} = E - \sum_{p=1}^{\infty} \alpha_p A^p \tag{A.2}$$

The operator A^p is a convolution operator, too, and has the matrix elements $a_{k-j}^{(p)}$ having the estimate

$$|a_{k-j}^{(p)}| \leqslant (C\beta)^p (\tfrac{3}{4})^{|k-j|}, \qquad k, j \in \mathbb{Z}^d$$

From this estimate and the decomposition (A.2) we come to the assertion of the lemma for $D_\beta^{1/2}$. In a similar way we get the assertion about $D_\beta^{-1/2}$.

A.3. The Proof of Lemma 5.1

From Lemma 4.1 and Remark 4.1 it follows easily that the matrix elements of the operator TS: $L_1^+ \to L_1^+$ have the estimate

$$|(TS)_{k-j}| \leqslant \mathrm{const} \cdot \beta^2 (\tfrac{1}{2})^{|k-j|}, \qquad k, j \in \mathbb{Z}^d$$

Then, as above, we get

$$(E_1 - TS)^{-1} e_{\delta_{k_0}} = e_{\delta_{k_0}} + \textstyle\sum_k g_{k_0-k} e_{\delta_k} \tag{A.3}$$

where

$$|g_j| \leqslant \mathrm{const} \cdot \beta^2 (\tfrac{3}{4})^{|j|}, \qquad j \in \mathbb{Z}^d$$

From this we obtain

$$h_{k_0,1}^+ = v_{k_0} + \sum_k g_{k_0-k} v_k$$

Because

$$v_k = \sum_j d_{k-j}^{(1/2)} w_j = w_k + \sum_j b_{k-j}^+ w_j, \qquad k \in \mathbb{Z}^d$$

we have

$$h_{k_0,1}^+ = w_{k_0} + \sum_k b_{k_0-k}^+ w_k + \sum_k g_{k_0-k} w_k + \sum_{k,j} g_{k_0-k} b_{k-j}^+ w_j$$

This gives (in the notations of Lemma 5.1)

$$\hat{q}_k^{(k_0)} = b_{k_0-k}^+ + g_{k_0-k} + (g * b^+)_{k_0-k} \tag{A.4}$$

The estimates (A.3) and (A.4) give the required estimate (5.2). ∎

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REFERENCES

1. S. Albeverio, Yu. G. Kondratiev, and M. Röckner, A remark on stochastic dynamics on the infinite-dimensional torus, in *Proceedings of the Ascona Conference 1993*, E. Bolthausen *et al.* (eds.) (Birkhäuser, 1995).
2. S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Uniqueness of the stochastic dynamics for continuous spin systems on a lattice, *J. Funct. Anal.* **133**:10–20 (1995).
3. S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Infinite dimensional diffusions, Markov fields, quantum fields and stochastic quantization, in *Stochastic Analysis and Applications in Physics*, A. I. Cardoso *et al.* (eds.) (Kluwer, Dordrecht, 1994).
4. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral Methods in Infinite Dimensional Analysis* (Kluwer, Dordrecht, 1995).
5. J. Bricmont, J. R. Fontaine, and L. J. Landau, On the uniqueness of the equilibrium state for plane rotators, *Commun. Math. Phys.* **56**:281–296 (1977).
6. R. L. Dobrushin, Markov processes with a large number of locally interacted components —Existence of the limiting process and its ergodicity, *Probl. Information Transmission* 7:70–87 (1971).
7. R. Holley, Recent results on the stochastic Ising model, *Rocky Mountain J. Math.* **4**:479–496 (1974).
8. R. Holley and D. Stroock, L_2 theory for the stochastic Ising model, *Z. Wahrsch. Verw. Gebiete* **35**:87–101 (1976).
9. R. Holley and D. Stroock, Applications of the stochastic Ising model to the Gibbs states, *Commun. Math. Phys.* **48**:249–265 (1976).
10. R. Holley and D. Stroock, Diffusions on an infinite dimensional torus, *J. Funct. Anal.* **42**:29–63 (1981).
11. Yu. G. Kondratiev, The construction of dynamics of the quantum lattice systems, *Sov. Math. Doklady* **282**:269–273 (1985).
12. F. Martinelli and E. Olivieri, Approach to equilibrium of Glauber dynamics in the one phase region, *Commun. Math. Phys.* **161**:447–486 (1994).
13. V. A. Malyshev and R. A. Minlos, *Gibbsian Random Fields* (Kluwer, Dordrecht, 1991).
14. V. A. Malyshev and R. A. Minlos, *Linear Infinite-Particle Operators* (American Mathematical Society, Providence, Rhode Island, 1995).
15. R. A. Minlos, Invariant subspaces of Ising stochastic dynamics, *Markov Processes Relat. Fields* **2**:263–284 (1996).
16. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. III: *Scattering Theory* (Academic Press, New York, 1979).
17. J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View* (Springer-Verlag, Berlin, 1987).